Extended one-step methods for solving delay-differential equations

F. Ibrahim1,2, A. A. Salama3, A. Ouazzi1 and S. Turek1

1 Institut für Angewandte Mathematik, LS III, TU Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany
2 South Valley University, Egypt
3 Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

Abstract: We discuss extended one-step methods of order three for the numerical solution of delay-differential equations. A convergence theorem and the numerical studies regarding the convergence factor of these methods are given. Also, we investigate the stability properties of these methods. The results of the theoretical studies are illustrated by numerical examples.

Keywords: Delay-differential equations, Stability, Convergence, Numerical solutions

1 Introduction

Delay differential equations (DDEs) have a wide range of applications in science and engineering. They arise in models where the rate of change of process is not only determined by its present state, but also by a certain past state. As for instance, the natural delay in the population mechanism. Bellen and Zennaro [4] developed a class of numerical methods to approximate the solution of DDEs. These methods are based on implicit Runge-Kutta methods. Paul and Baker [19] used explicit Runge-Kutta methods. The retarded argument is approximated by an appropriate Hermite interpolation. The same methods are used by Arndt [2] with a different step size control mechanism. Bellen and Zennaro [4] developed a class of numerical methods for the computation of periodic solution of DDEs. Hu and Cahlon [12] considered continuous Runge-Kutta methods for the numerical solutions of retarded and neutral DDEs. Engelborghs et al. [6] presented collocation methods for the computation of periodic solution of DDEs. Hu and Cahlon [12] considered the numerical solution of initial-value discrete-delay systems.

The most obvious of the above methods for solving problem (1) numerically are \( \theta \)-methods given in the following form

\[
y_{n+1} = y_n + h[(1-\theta)f_n + \theta f_{n+1}], \quad n = 0, \ldots, N-1.
\]

where \( \theta \) is a parameter set to be \( 0 \leq \theta \leq 1 \), \( N \) is the number of nodes, \( h \) is the uniform step length and \( y_n \) is an approximation to the exact solution \( y(x_n) \) at the mesh point \( x_n = a + nh \). Furthermore,

\[
f_n = f(x_n, y_n, y(\alpha(x_n)))
\]

\[\text{H1: The Lipschitz condition holds:}
|f(x, y_1, z_1) - f(x, y_2, z_2)| \leq L_1 |y_1 - y_2| + L_2 |z_1 - z_2|.
\]

where \( L_1 \) and \( L_2 \) are Lipschitz constants.

\[\text{H2: For any } y \in C^1[a, b], \
\text{the mapping } x \rightarrow f(x, y(x), y(\alpha(x))) \text{ is continuous.}
\]

Under the conditions H1 and H2, the problem (1, 2) has a unique solution Driver (see [7]).

Many methods have been proposed for the numerical approximation of problem (1, 2). Oberle and Pesch [18] introduced a class of numerical methods for the treatment of DDEs based on the well-known Runge-Kutta-Fehlberg methods. The retarded argument is approximated by an appropriate Hermite interpolation. The same methods are used by Arndt [2] with a different step size control mechanism. Bellen and Zennaro [4] developed a class of numerical methods to approximate the solution of DDEs. These methods are based on implicit Runge-Kutta methods. Paul and Baker [19] used explicit Runge-Kutta methods. The retarded argument is approximated by an appropriate Hermite interpolation. The same methods are used by Arndt [2] with a different step size control mechanism. Bellen and Zennaro [4] developed a class of numerical methods to approximate the solution of DDEs. These methods are based on implicit Runge-Kutta methods. Paul and Baker [19] used explicit Runge-Kutta methods.
where \( y_h(x) = g(x) \) for \( x \leq a \) and \( y_h(x) \) with \( x > a \) is defined by piecewise linear interpolation, i.e.

\[
y^h(x) = \frac{x_{k+1} - x}{h} y_k + \frac{x - x_k}{h} y_{k+1}, \quad \text{for} \quad x_k \leq x \leq x_{k+1}; \quad k = 0, 1, \ldots
\]

(6)

In general the \( \theta \)-methods described by (4), (6) and (5) are of first order and they are at most of second order for \( \theta \) set to equal 0.5. The stability of \( \theta \)-methods has been considered with respect to the following linear DDEs

\[
y(x) = \lambda y(x) + \mu y(x - \tau), \quad x \geq 0
\]

\[
y(x) = g(x), \quad -\tau \leq x \leq 0
\]

(7)

where \( \lambda \) and \( \mu \) are complex numbers and \( \tau > 0 \). It is known, see Al-Mutib [1], that under the following two conditions

1. \( g(x) \) is continuous
2. \( P \)-stability \( |\mu| < -Re(\lambda) \),

the solution \( y(x) \) of linear DDEs (7) tends to zero as \( x \) tends to infinity. The adaptation of \( \theta \)-methods has already been considered in the literature, Calvo and Grande [5], Liu and Spijker [16], In’t Hout and Spijker [13], Guglielmi [8] and Van Den Heuvel [21] and [22].

Our work aims to extend the current \( \theta \)-methods to be of third order. Moreover, as these methods depend on a free parameter, we determine the range of the free parameter which guarantees the stability of these methods. The paper is organized as follows: In the next Section, we derive our third order methods for solving DDEs. Section 3 is devoted to the investigation the stability of the methods and the determination of the stability regions. In Section 4, we determine the convergence factor of the present methods. The proof of convergence of the present methods is given in Section 5. Finally, in Section 6, we test numerically our extended methods and make numerical comparison with the \( \theta \)-methods.

2 Extended one-step third-order methods

In this section, we extend the work of Usmani and Agarwal [23], Jacques [14] and Kondrat and Jacques [15] to derive the present methods. We start with the following discretization for the numerical solution of (1)

\[
y_{n+1} = y_n + h [\alpha_0 f_n + \alpha_1 f_{n+1} + \alpha_2 \hat{f}_{n+2}, \quad n = 0, 1, \ldots, N - 1
\]

(8)

where

\[
\hat{f}_{n+2} = f(x_{n+2}, \hat{y}_{n+2}, y_h(\alpha(x_{n+2})))
\]

(9)

In order to determine the coefficients \( \alpha_0, \alpha_1 \) and \( \alpha_2 \), we rewrite (8) in the exact form

\[
y(x_{n+1}) = y(x_n) + h [\alpha_0 f(x_n, y(x_n), y(\alpha(x_n)))
\]

\[
+ \alpha_1 f(x_{n+1}, y(x_{n+1}), y(\alpha(x_{n+1})))
\]

\[
+ \alpha_2 f(x_{n+2}, y(x_{n+2}), y(\alpha(x_{n+2})))) + t(x_{n+1}).
\]

(10)

We expand the left and right sides of (10) in the Taylor series at point \( x_{n+1} \), equate the coefficients up to the third order terms \( O(h^3) \) and solving the resulting system of equations, we obtain

\[
\alpha_0 = \frac{5}{12}, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = \frac{1}{12}
\]

(11)

and

\[
t(x_{n+1}) = \frac{h^3}{24} \beta_0(4)(\xi)
\]

(12)

where \( x_n < \xi < x_{n+2} \). Substituting the alpha coefficients from (11) into (8), we obtain

\[
y_{n+1} = y_n + \frac{h}{12} [5 f_n + 8 f_{n+1} - f_{n+2}]
\]

(13)

Here, and in the following

\[
y^h(x) = g(x) \quad \text{for} \quad x \leq a
\]

and \( y^h(x) \) with \( x > a \) is defined by

\[
y^h(x) = \beta_0 y_k + \beta_1 y_{k+1}
\]

\[
+ h [\beta_2 f_k + \beta_3 f_{k+1}], \quad \text{for} \quad x_k < x \leq x_{k+1}; \quad k = 0, 1, \ldots
\]

(15)

In order to determine the coefficients \( \beta_0, \beta_1, \beta_2 \) and \( \beta_3 \), we rewrite (15) in the exact form

\[
y(x) = \beta_0 y_k + \beta_1 y_{k+1}
\]

\[
+ h [\beta_2 f_k + \beta_3 f_{k+1}, \quad y(x_k), y(\alpha(x_k))]
\]

\[
+ \beta_4 f(x_{k+1}, y(x_{k+1}), y(\alpha(x_{k+1})))) + t(x_{k+1})
\]

(16)

Similarly, we expand the left and right sides of (16) with Taylor series at point \( x_{k+1} \) and equate the coefficients up to the terms of second order \( O(h^3) \). We obtain the resulting system of equations

\[
\begin{cases}
\beta_0 + \beta_1 = 1 \\
\beta_0 - \beta_2 - \beta_3 = -\delta(x) \\
\beta_0 - 2\beta_2 = \delta^2(x)
\end{cases}
\]

(17)

where

\[
\delta(x) = \frac{1}{h} (x - x_{k+1})
\]

(18)

The solution of the above system (17) is

\[
\begin{cases}
\beta_0 = 1 - \beta_1 \\
\beta_2 = \frac{1}{2}(1 - \beta_1 - \delta^2(x)) \\
\beta_3 = \frac{1}{2}(\delta^2(x) + 2\delta(x) - \beta_1 + 1)
\end{cases}
\]

(19)
In order to solve the Problem (7), the present methods are based on the characteristic polynomial associated with (24) takes the form

\[
W_m(z) = \left[24 - 2X(8 - \beta_1) + X^2(4 - \beta_1)\right] z^{m+1} \\
- \left[24 + 2X(4 + \beta_1) + X^2\beta_1\right] z^m \\
- Y \left[10 + X\beta_1 + (16 - X(4 - \beta_1))z - 2z^2\right] \\
= 0, \quad m = 1, 2, \ldots
\]

It is clear that \((X, Y) \in S_P\) if and only if all roots of the polynomials \(W_m\) are inside the unit disc for \(m = 1, 2, \ldots\). Let

\[
P(z) := \left[24 - 2X(8 - \beta_1) + X^2(4 - \beta_1)\right] z^{m+1} \\
- \left[24 + 2X(4 + \beta_1) + X^2\beta_1\right] z^m, \\
Q(z) := -Y \left[10 + X\beta_1 + (16 - X(4 - \beta_1))z - 2z^2\right]
\]

and \(z^*\) denotes the only nonzero root of \(P(z)\). It follows from Rouche’s theorem, see Marden [17], that \((X, Y) \in S_P\) if \(\lvert z^* \rvert < 1\) and \(\lvert P(z) \rvert > \lvert Q(z) \rvert\) on the unit circle. Furthermore, on the unit circle we have

\[
\lvert P(z) \rvert \geq \left[\left\lvert 24 - 2X(8 - \beta_1) + X^2(4 - \beta_1)\right\rvert \right] z^{m+1} \\
- \left[\left\lvert 24 + 2X(4 + \beta_1) + X^2\beta_1\right\rvert \right] z^m, \\
\lvert Q(z) \rvert \leq \left\lvert Y \left[10 + X\beta_1\right] \right\rvert \\
+ \left\lvert 16 - X(4 - \beta_1) \right\rvert + 2).
\]

Therefore, \((X, Y) \in S_P\) if the following set of inequalities are satisfied

\[
\left[\left\lvert 24 - 2X(8 - \beta_1) + X^2(4 - \beta_1)\right\rvert \right] z^{m+1} \\
- \left[\left\lvert 24 + 2X(4 + \beta_1) + X^2\beta_1\right\rvert \right] z^m \geq \left\lvert Y \left[10 + X\beta_1\right] \right\rvert \\
+ \left\lvert 16 - X(4 - \beta_1) \right\rvert + 2),
\]

and

\[
\left[\frac{24 + 2X(4 + \beta_1) + X^2\beta_1}{24 - 2X(8 - \beta_1) + X^2(4 - \beta_1)}\right] < 1.
\]

It can be seen that \(X \in S_A\) where \(S_A\) is the \(A\)-stability region of the present methods for solving ordinary differential equation if and only if (29) is satisfied, we refer to Hairer et al. [9] for more details concerning the \(A\)-stability concept. It is easy to see that (29) is satisfied if \(\beta_1 \in (0, 2)\). Moreover, the \(P\)-stability region for various values of \(\beta_1 \in (0, 2)\) is determined by solving the system of inequalities (28) and (29). Thus we establish the following.

**Theorem 1.** For the present methods, the region of \(P\)-stability satisfies the relation

\[
S_P \cap R = \{ (X, Y) : Y < -X \text{ and } |Y| < \phi(X) \}
\]

where

\[
\phi(X) = \begin{cases} 
\frac{X^2 - 6X}{7 - X} & \text{for } X \geq -3 \\
\frac{X^2 - 2X + 12}{7 - X} & \text{for } X < -3
\end{cases}
\]
for $\bar{\beta}_1 = 0$.

Proof. The proof follows immediately from inequality (28).

Theorem 2. For the present methods the region of $P$-stability satisfy the relation

$$S_P \cap R = \{(X, Y) : Y < -X and |Y| < \phi(X)\},$$

where

$$\phi(X) = \begin{cases} \frac{(2-\beta_1)X^2 - 12X}{14 - (2-\beta_1)X}, & \text{if } X \geq -\frac{10}{\beta_1}, \\ \frac{(2-\beta_1)X^2 - 12X}{2(2-X)}, & \text{if } X < -\frac{10}{\beta_1}, \end{cases}$$

for $\bar{\beta}_1 \in (0, 2]$.

Proof. The proof follows immediately from inequality (28).

Theorem 3. For the present methods the region of $P$-stability satisfy the relation

$$S_P \cap R = \{(x, y) : |Y| < -X and Y < \phi(X)\},$$

where

$$\phi(X) = \begin{cases} \frac{(\beta_1 - 2)X^2 - 12X}{(1-\beta_1)X - 14}, & \text{if } X > -\left(\frac{\beta_1 + 4}{\beta_1}\right), \\ -\sqrt{\left(\frac{\beta_1 + 4}{\beta_1}\right)^2 - \frac{24}{\beta_1}}, & \text{if } X < -\left(\frac{\beta_1 + 4}{\beta_1}\right) \right), \\ \frac{2X^2 - (1-2\beta_1)X + 24}{14 - X(1-\beta_1)}, & \text{if } X < -\left(\frac{\beta_1 + 4}{\beta_1}\right), \\ -\sqrt{\left(\frac{\beta_1 + 4}{\beta_1}\right)^2 - \frac{24}{\beta_1}}, & \text{if } X < -\left(\frac{\beta_1 + 4}{\beta_1}\right) \right), \end{cases}$$

for $\bar{\beta}_1 \in (-\infty, 0)$.

Proof. The proof follows immediately from inequality (28).

The Fig. 1 shows the different regions of the $P$-stability with respect to different values of $\bar{\beta}_1$.

4 Convergence factor for the present methods

In this section, we present the main result concerning the convergence factor of the methods (13), (21) and (22) with $\bar{\beta}_1 = 0$. This case may be expressed in the form

$$y_{n+1}^{(j+1)} = y_n + \frac{h}{12} \left[ 5f_n + 8f(x_{n+1}, y_{n+1}^{(j)}, y_{n+1}^{(j)}(\alpha(x_{n+1}))) \\ - f(x_{n+2}, y_{n+2}^{(j)}, y_{n+2}^{(j)}(\alpha(x_{n+2}))) \right], \quad j = 1, \ldots, \quad y_{n+1}^{(j)}(x) = y_k + \frac{h}{2} \left[ (1 - \delta^2(x))f_k + (1 + \delta(x))^2 f_{k+1} \right],$$

for $x_k < x \leq x_{k+1}$; $k = 0, 1, \ldots$ \hspace{1cm} (30)

where $y_{n+1}^{(0)}$ is an initial approximation to the solution $y$ at $x_{n+1}$ and $y_{n+1}^{(j)}$, $j \geq 1$ are Picard iterations.

Now, we state and prove the following theorem.

Theorem 4. If the sequence $(y_{n+1}^{(j)})$ given by (30) is bounded by a constant $C$ and the condition

$$hL < \frac{-2R_1 + 2\sqrt{R_1^2 + 6R_2}}{R_2}$$

$$R_1 = 4 + 3r_1^2$$

$$R_2 = r_2^2 + 2r_2 + 5$$

is satisfied, where $r_1, r_2 \in (0, 1)$ and $L = \max\{L_1, L_2\}$. Then, the method (30) is convergent.
Moreover, if $\alpha(x_{n+1}) \in (x_n, x_{n+1})$, we have
\[
y^h(\alpha(x_{n+1})) = y_k + \frac{h}{2} \left[ (1 - \delta^2(\alpha(x_{n+1})))f_k + (1 + \delta(\alpha(x_{n+1})))^2 f_{k+1} \right]
\]
(32)

and
\[
y^h(\alpha(x_{n+1})) = y_n + \frac{h}{2} \left[ (1 - \delta^2(\alpha(x_{n+1})))f_n + (1 + \delta(\alpha(x_{n+1})))^2 f_{n+1} \right]
\]
(33)

Since $\alpha(x_{n+1}) - x_n \leq h$, we let $\alpha(x_{n+1}) - x_n = r_j h$ with $r_j \in (0, 1]$. Then, from (33) and (34) we obtain
\[
y^{h,j}(\alpha(x_{n+1})) = y_n + \frac{h}{2} \left[ (1 - \delta^2(\alpha(x_{n+1})))f_n + (1 + \delta(\alpha(x_{n+1})))^2 f_{n+1} \right]
\]
(34)

Using the Lipschitz condition, it follows that
\[
\left| y^{h,j}(\alpha(x_{n+1})) - y^h(\alpha(x_{n+1})) \right| \leq \frac{hl^2}{2} \left| x_{n+1} - x_n \right|.
\]
(35)

Similarly, let $\alpha(x_{n+2}) - x_{n+1} = r_j h$ with $r_j \in (0, 1]$, we get
\[
\left| y^{h,j}(\alpha(x_{n+2})) - y^h(\alpha(x_{n+2})) \right| \leq \frac{hl^2}{2} \left| x_{n+2} - x_{n+1} \right|.
\]
(36)

From (37), it follows
\[
\left| y^{(j)}_{n+2} - y_{n+2} \right| \leq \frac{4hl^2}{2} \left| x_{n+1} - x_n \right|.
\]
(38)

Using (36), (37) and (38), we obtain
\[
\left| y^{(j)}_{n+1} - y_{n+1} \right| \leq C \left| x_{n+1} - x_n \right|, \quad j = 0, 1, \ldots
\]
(39)

where
\[
C = \frac{hL(16 + 4hl + hL(1 + r_j)^2)}{12(2 - hl^2)}.
\]

The constant $C$ is referred as the convergence factor. Thus, the iterative process (30) is convergent if $C < 1$, or if $hl$ satisfies the condition (31). This completes proof of the theorem.

**Remark.** In the same manner, one can determine the convergence factor for different values of $\beta_1$.

### 5 Error estimate

We state and prove the error estimate for the methods (13), (21) and (22). Our error estimate is given by the following theorem:

**Theorem 5.** Let $y_n$ be obtained by the methods (13), (21) and (22). Then, at each mesh point $x_n$, we have the following error estimate:
\[
e_n = |y(x_n) - y_n| \leq C_1 h^n, \quad n = 1, 2, \ldots, N
\]
(40)

where $C_1$ is independent of $n$ and $h$.

**Proof.** Without loose of generality, we take $\beta_1 = 0$. Subtracting (13) from (10) with the coefficients in (11), we obtain
\[
y(x_{n+1}) - y_{n+1} = y(x_n) - y_n + \frac{h}{12} \left[ f(x_{n+1}, y_{n+1}) - f(x_n, y_n - h/2) \right] + O(h^2).
\]
(41)

From the definition of $e_n$ in (40) and the Lipschitz condition (3), we obtain
\[
e_{n+1} \leq e_n + \frac{h}{12} \left[ 5(L_1 e_n + L_2 e_{n+1}) + 8(L_1 e_{n+1} + L_2 e_{n+2}) + L_1 \hat{e}_{n+2} + L_2 \hat{e}_{n+2} \right] + O(h^4).
\]
(42)

where
\[
e_{\alpha_j} = \left| y(\alpha(x_n)) - y^\alpha(\alpha(x_n)) \right|
\]
(43)

and
\[
\hat{e}_{n+2} = \left| y(x_{n+2}) - \hat{y}_{n+2} \right|.
\]
(44)

Form (15), the inequality (42) can be rewritten as follows
\[
e_{n+1} \leq e_n + \frac{h}{12} \left[ 5(L_1 e_n + L_2 e_{\alpha_j}) + 8(L_1 e_{n+1} + L_2 e_{\alpha_{j+1}}) + L_1 \hat{e}_{n+2} + L_2 \hat{e}_{n+2} \right] + O(h^4).
\]
(45)

Now, we estimates the quantities $e_{\alpha_j}, e_{\alpha_{j+1}}, e_{\alpha_{j+2}}$ and $\hat{e}_{n+2}$ in (42). From (16) and (21) with the coefficient in (19), we obtain
\[
e_{\alpha_j} \leq e_k + g_1(x_k)(L_3 e_k + L_2 e_{\alpha_j}) + g_2(x_k)(L_1 e_{\alpha_j+1} + L_2 e_{\alpha_{j+1}}) + \hat{L}(x_{k+1}),
\]
(46)

for $x_k < \alpha(x_n) < x_{k+1}$; $k = 0, 1, \ldots$
where
\[ g_1(x) = \frac{h}{2}(1 - \delta^2(\alpha(x))) \] (47)
and
\[ g_2(x) = \frac{h}{2}(1 + \delta(\alpha(x)))^2. \] (48)

Let us consider
\[ E_n = \max_{0 \leq k \leq n} e_k \]
and
\[ E_{\alpha_n} = \max_{0 \leq k \leq \alpha_n} e_k. \]

Then (46) with (20) leads to the following estimation
\[ E_{\alpha_{n+1}} \leq E_n + g_1(x_{n+1})(L_1E_n + L_2E_{\alpha_n}) \]
\[ + g_2(x_{n+1})(L_1E_{n+1} + L_2E_{\alpha_{n+1}}) \]
\[ + O(h^3); \quad j = 0, 1, 2 \]
and
\[ \bar{E}_{n+2} \leq E_n + 2h(L_1E_n + L_2E_{\alpha_{n+1}}) + O(h^3), \] (50)
where
\[ \bar{E}_{n+2} = \max_{0 \leq k \leq n+2} \bar{e}_k. \]

Assume that \( h \) is sufficiently small to satisfy \( g_1(x)L_2 < 1 \) and \( g_2(x)L_2 < 1 \). Using (49), we rewrite \( E_{\alpha_{n+j}} \) in terms of \( E_{n+j} \), for \( j = 0, 1 \), as follows
\[ E_{\alpha_n} \leq \frac{1}{g_3(x_n)} (g_4(x_n)E_n + g_2(x_n)L_1E_{n+1}) + O(h^3) \] (51)
and
\[ E_{\alpha_{n+1}} \leq \frac{1}{g_3(x_n)} (g_5(x_n)E_n + g_6(x_n)E_{n+1}) + O(h^3), \] (52)
where
\[ g_3(x) = 1 + g_1(x)L_2 - g_2(x+h)L_2 \]
\[ + g_1(x)g_2(x+h)L_2 - g_1(x+h)g_2(x)L_2, \]
\[ g_4(x) = 1 + g_1(x)L_2 - g_2(x+h)L_2 \]
\[ + g_1(x+h)g_2(x)L_2, \]
\[ g_5(x) = 1 - g_1(x)L_2 + g_1(x+h)L_1 + L_2, \]
\[ g_6(x) = g_2(x+h)L_1 - g_1(x)g_2(x+h)L_1 \]
\[ + g_1(x+h)g_2(x)L_1. \]

Substituting (51) and (52) into (49) for \( j = 2 \), we get
\[ E_{\alpha_{n+2}} \leq (1 + g_1(x_{n+2})L_1)E_n + g_2(x_{n+2})L_1E_{n+1} \]
\[ + \frac{L_2}{g_3(x_n)} (g_4(x_n)g_1(x_{n+2})E_n \]
\[ + g_5(x_n)g_2(x_{n+2})E_n \]
\[ + g_6(x_n)g_1(x_{n+2})L_1E_{n+1} \]
\[ + g_6(x_n)g_2(x_{n+2})E_{n+1} \] + \( O(h^3) \), (54)
and
\[ \bar{E}_{n+2} \leq E_n + 2hL_1E_{n+1} \]
\[ + \frac{2hL_2}{g_3(x_n)} [g_5(x_n)E_n + g_6(x_n)E_{n+1}] + O(h^3). \] (55)

Using (49-52) and (42), we obtain
\[ E_{n+1} \leq (1 + h\tilde{g}_1(x_n))E_n + h\tilde{g}_2(x_n)E_{n+1} + B h^4, \] (56)
where \( B \) is a nonnegative constant and the \( \tilde{g}_1 \) and \( \tilde{g}_2 \) are defined as follows
\[ \tilde{g}_1(x) = \frac{1}{12} [6L_1 + L_2 + L_1L_2g_1(x + 2h)] \]
\[ + \frac{1}{12g_3(x)} [(5 + g_1(x + 2h)L_2)g_4(x)L_2 \]
\[ + (8 + 2hL_1 + g_2(x + 2h)L_2)g_5(x)L_2], \]
\[ \tilde{g}_2(x) = \frac{1}{12} [8L_1 + 2hL_2^2 + g_2(x + 2h)L_1L_2 \]
\[ + \frac{1}{12g_3(x)} [(5 + g_1(x + 2h)L_2^2)g_2(x)L_1 \]
\[ + (8 + 2hL_1)g_6(x)L_2]. \] (57)

Thus, (56) can be rewritten as
\[ E_{n+1} \leq \frac{1}{1 - h\tilde{g}_2(x_n)} E_n \]
\[ + \frac{B}{1 - h\tilde{g}_2(x_n)} h^4. \] (58)

Assume that \( h \) is sufficiently small to ensure \( h\tilde{g}_2(x) < 1 \). Then, there exists two positive constants \( w_1 \) and \( w_2 \) such that
\[ \frac{1 + h\tilde{g}_1(x_n)}{1 - h\tilde{g}_2(x_n)} \leq 1 + hw_1, \]
\[ \frac{h^4}{1 - h\tilde{g}_2(x_n)} \leq h^4 w_2. \] (59)

Then,
\[ E_{n+1} \leq (1 + hw_1)E_n + Bw_2 h^4. \] (60)
Applying Henrici Lemma [11] to the inequality (60) yields
\[ E_n \leq E_0 + \frac{h^3}{w_1} w_2 B e^{hw_1} - 1. \] (61)
Since \( E_0 = 0 \), then
\[ E_n \leq \frac{h^3}{w_1} w_2 B (e^{hw_1} - 1). \] (62)

This completes the proof of Theorem 5.

### 6 Numerical tests

In this section, we validate our methods (13), (21) and (22) numerically for different values of \( \beta_1 \). The comparison with \( \theta \)-methods for different values of \( \theta \) is
considered as well. We restrict our study to equidistant steps size time.

**Example 1**

\[
y'(x) = \frac{1}{2} e^x y\left(\frac{x}{2}\right) + \frac{x}{2} y(x), \quad 0 \leq x \leq 1
\]

\[
y(0) = 1
\]

The exact solution is \( y(x) = e^x \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \theta = 0 )</th>
<th>( \theta = 0.5 )</th>
<th>( \theta = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>( E^N )</td>
<td>( R^N )</td>
<td>( E^N )</td>
</tr>
<tr>
<td>10</td>
<td>5.83E-02</td>
<td>6.93E-04</td>
<td>6.28E-02</td>
</tr>
<tr>
<td>20</td>
<td>2.97E-02</td>
<td>0.97</td>
<td>1.73E-04</td>
</tr>
<tr>
<td>40</td>
<td>1.50E-02</td>
<td>0.98</td>
<td>4.33E-05</td>
</tr>
<tr>
<td>80</td>
<td>7.52E-03</td>
<td>0.99</td>
<td>1.08E-05</td>
</tr>
<tr>
<td>160</td>
<td>3.77E-03</td>
<td>0.99</td>
<td>7.59E-03</td>
</tr>
</tbody>
</table>

**Table 1** Comparison of Extended one-step method with parameter \( \beta \) set to equal 0 and 1 with the \( \theta \)-methods with \( \theta \) set to equal 0, 0.5 and 1 for Example 1.

**Example 2**

\[
y'(x) = y_1(x - 1) + y_2(x), \quad x \geq 0
\]

\[
y_2(x) = y_1(x) - y_1(x - 1)
\]

\[
y_1(x) = e^x, \quad x \leq 0
\]

\[
y_2(0) = 1 - e^{-1}
\]

The exact solution is

\[
y_1(x) = e^x, \quad x \geq 0
\]

\[
y_2(x) = 1 - e^{-x-1}, \quad x \geq 0
\]

The tables **Table 1** and **Table 2** show the error reduction, \( E^N \), with respect to time step size refinement, \( h = 1/N \), and the rate order of convergence, \( R^N \), for the \( \theta \)-methods as well as our extended step-one methods. All examples confirm the theoretical studies introduced in this paper, mainly the third order of convergence of our extended one-step methods.

**7 Conclusion and perspective**

We have introduced a new numerical methods of third order for solving delay differential equations. The

\[
\theta\text{-methods}
\]

\[
y_1(x)
\]

\[
y_2(x)
\]

\[
N \quad E^N \quad R^N \quad E^N \quad R^N \quad E^N \quad R^N
\]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( E^N )</th>
<th>( R^N )</th>
<th>( E^N )</th>
<th>( R^N )</th>
<th>( E^N )</th>
<th>( R^N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.12E-02</td>
<td>8.93E-03</td>
<td>2.00</td>
<td>3.08E-02</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>2.80E-03</td>
<td>2.00</td>
<td>3.08E-03</td>
<td>2.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>7.00E-04</td>
<td>2.00</td>
<td>5.57E-04</td>
<td>2.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>1.75E-04</td>
<td>2.00</td>
<td>1.39E-04</td>
<td>2.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>4.37E-05</td>
<td>2.00</td>
<td>3.48E-05</td>
<td>2.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2** Comparison of Extended one-step method with \( \theta \)-methods with \( \theta \) set to equal 0, 0.5 and 1 for Example 2.

\( P \)-stability region has been investigated for different values of parameter \( \beta \in (-\infty, 2] \). We showed that the larger \( P \)-stability region occurs for \( \beta = 0 \). Moreover, the methods are L-stable for solving ODEs for the case corresponding to \( \beta = 0 \). The effectiveness of our methods is clearly indicated with the numerical results. Our research perspective is to extend the current study for integral-deferential equations.

**References**


